Chapter 10
Inventory Theory

10.1. (a) Find the smallest n such that $\Delta g(n) \geq 0$.

$$\Delta g(1) = -3$$
$$\Delta g(2) = 2$$

$$n^* = 2$$

(b) Find the smallest n such that $\Delta g(n) \leq 0$.

$$\Delta g(1) = \frac{1}{25} - \frac{1}{64}$$
$$\Delta g(2) = \frac{1}{4} - \frac{1}{25}$$
$$\Delta g(3) = 1 - \frac{1}{4}$$
$$\Delta g(4) = \frac{1}{16} - 1 < 0$$

$$n^* = 4$$

(c) Find the smallest n such that $\Delta g(n) \geq 0$.

$$\Delta g(1) = 0.357$$
$$\Delta g(2) = 0.122$$
$$\Delta g(3) = -0.015$$
$$n^* = 3$$

10.3. (a) Using Eq. 7.2, q^* is the smallest q such that:

$$F(q) \geq \frac{445}{445 + 350} = 0.55975$$

and

$$\begin{aligned} F(0) &= 0.0067 \\ F(1) &= 0.0404 \\ F(2) &= 0.1247 \\ F(3) &= 0.2650 \\ F(4) &= 0.4405 \\ F(5) &= 0.6160 \end{aligned}$$

Because $F(5) = 0.6160 > 0.55975$, then $q^* =$ order up to 5 computers.
(b) Order up to 51 computers.

10.5. Order up to 68 cameras.

10.7. Let $c_r =$ the retail price of the item in inventory. Then, the expected profit per period is given by

$$E[P_{x,y}] = c_r \sum_{k=0}^{y} k f_k - E[C_{x,y}]$$

To maximize profits, we use difference equations as follows:

$$\begin{aligned} \Delta_y E[P_{x,y}] &= c_r \sum_{k=0}^{y+1} k f_k - \Delta_y E[C_{x,y}] - c_r \sum_{k=0}^{y} k f_k \\ & c_r(y+1) f_{y+1} - \Delta_y E[C_{x,y}] \leq 0 \\ &\text{or} \\ \Delta_y E[C_{x,y}] &\geq c_r(y+1) f_{y+1} \end{aligned}$$

This problem can also be solved under the assumption that all items are sold in this period or the next period, then

$$E[P_{x,y}] = c_r \sum_{k=0}^{\infty} k f_k - E[C_{x,y}]$$

and

$$\begin{aligned} \Delta_y E[P_{x,y}] &= -\Delta_y E[C_{x,y}] \leq 0 \\ &\text{or} \\ \Delta_y E[C_{x,y}] &\geq 0 \end{aligned}$$

10.9. The cost of ordering up to S is:

$$\begin{aligned} K + c_w S + L(S) &= 170 + 400 \times 68 + L(68) \\ &= 170 + 27200 + 3159.60 \\ &= 30529.60 \end{aligned}$$

The reorder point, s, is the solution to the following:

$$30529.60 = 400 \times s + L(s)$$

Using trial and error, the integer value of s that solves the equation is $s = 65$. Thus, the optimal policy calls for reordering up to 68 cameras whenever the end-of-period inventory is 65 cameras or less.

10.11. From example 7.3 the optimal policy is to order up to 257. However, assuming that the discount is taken, the optimal order up to quantity is given by the smallest S such that

$$F(S) \geq \frac{c_p - c_{w_2}}{c_p + c_h} = \frac{25 - 4.5}{25 + 0.5} = 0.8039$$

which results in $S = 100 + 200 \times 0.8039 = 260.78$, then

$$K + c_w s + L(s) = 500 + 4.5(261 - 99) + L(261) = 1851.46375$$

The optimal reorder point, s, is the value that satisfies:

$$\begin{aligned} 1801.96 &= c_{w_2} s + L(s) \\ &= 4.5s + \frac{51}{800}s^2 - 37.75s + 5637.5 \end{aligned}$$

or, solving the quadratic equation

$$0 = \frac{51}{800}s^2 - 33.25s + 3835.54$$

The solution is given by,

$$s = \frac{33.25 \pm \sqrt{33.25^2 - 4(\frac{51}{800})(3835.54)}}{2(\frac{51}{800})}$$

$$s = \begin{cases} 349.35 \\ 172.22 \end{cases}$$

Because 172.22 is smaller than the optimal order quantity without setup cost, the optimal policy with setup cost is to order up to 261 widgets whenever the inventory from the previous period is less than 173. Thus, the price break does not change the optimal policy, because the difference $(S - s)$ is less than 100 units.

10.13. (a) Planning Horizon of 3 weeks. Here, we are trying to maximize profits, then

$$E[P_{x,y}] = \sum_{k=0}^{y} s_p k f_k - E[C_{x,y}]$$

where s_p is the selling price and $E[C_{x,y}]$ is given by Equation 7.5. The binomial distribution with parameters 4 and 0.4, has the following values: $f_0 = 0.1296$, $f_1 = 0.3456$, $f_2 = 0.3456$, $f_3 = 0.1536$, and $f_4 = 0.0256$.

$$\begin{aligned}
L(0) &= 0 \times f_0 &&= 0.000\\
L(1) &= 15 \times f_0 + 0 \times f_1 &&= 1.944\\
L(2) &= 30 \times f_0 + 15 \times f_1 + 0 \times f_2 &&= 9.072\\
L(3) &= 45 \times f_0 + 30 \times f_1 + 15 \times f_2 + 0 \times f_3 &&= 21.384\\
L(4) &= 60 \times f_0 + 45 \times f_1 + 30 \times f_2 + 15 \times f_3 + 0 \times f_4 &&= 36.000
\end{aligned}$$

Table: values for $E[P_{x,y}] = \sum_{k=0}^{y} s_p k f_k - c_w(y-x) - L(y)$. Production up to quantity for week 3

x	y=0	y=1	y=2	y=3	y=4
0	0+0	587.52 - 1400 - 1.94	1762.56 - 2000 - 9.07	2545.92 - 2500 - 21.38	2720 - 3250 - 36
1		587.52 - 0 - 1.94	1762.56 - 1400 - 9.07	2545.92 - 2000 - 21.38	2720 - 2500 - 36
2			1762.56 - 0 - 9.07	2545.92 - 1400 - 21.38	2720 - 2000 - 36
3				2545.92 - 0 - 21.38	2720 - 1400 - 36
4					2720 - 0 - 36

Optimal values for the end of the second week

x	0	1	2	3	4
v_3	24.54	585.58	1753.49	2524.54	2684.00
y_3	3	1	2	3	4

The decision made at the end of week 1 (i.e., for week 2's beginning inventory) is determined through the following equation:

$$v_2(x) = max_{y \geq x}\{E[P_{x,y}] + E_y[v_3(x_2)]\}$$

where the values of $E[P_{x,y}]$ are given in the table above, and $E_y[v_3(x_2)]$ are as follows:

$$\begin{aligned}
E_0[v_3(x_2)] &= 24.54 \times 1 &&= 24.54\\
E_1[v_3(x_2)] &= 585.58 \times .1296 + 24.54 \times .8704 &&= 97.25\\
E_2[v_3(x_2)] &= 1753.49 \times .1296 + 585.58 \times .3456 + 24.54 \times .5248 &&= 442.50\\
E_3[v_3(x_2)] &= 2524.54 \times .1296 + 1753.49 \times .3456 + \\
&\quad 585.58 \times .3456 + 24.54 \times .1792 &&= 1139.96\\
E_4[v_3(x_2)] &= 2884.00 \times .1296 + 2524.54 \times .3456 + \\
&\quad 1753.49 \times .3456 + 585.58 \times .1536 + \\
&\quad 24.54 \times .0256 &&= 1916.90
\end{aligned}$$

Values for $E[P_{x,y}] + E_y[v_3(x_2)]$. Production up to quantity for week 2

x	y=0	y=1	y=2	y=3	y=4
0	0 + 24.54	-814.42 + 97.25	-246.51 + 442.50	24.54 + 1139.96	-566.00 + 1916.90
1		585.58 + 97.25	353.49 + 442.50	524.54 + 1139.96	184.00 + 1916.90
2			1753.49 + 442.50	1124.54 + 1139.96	684.00 + 1916.90
3				2524.54 + 1139.96	1284.00 + 1916.90
4					2684.00 + 1916.90

Optimal values for the end of the first week:

x	0	1	2	3	4
v_2	1350.90	2100.90	2600.90	3664.49	4600.90
y_2	4	4	4	3	4

$$\begin{aligned}
E_0[v_2(x_1)] &= 1350.90 \times 1 && = 1350.90 \\
E_1[v_2(x_1)] &= 2100.90 \times .1296 + 1350.90 \times .8704 && = 1448.10 \\
E_2[v_2(x_1)] &= 2600.90 \times .1296 + 2100.90 \times .3456 + 1350.90 \times .5248 && = 1772.10 \\
E_3[v_2(x_1)] &= 3664.49 \times .1296 + 2600.90 \times .3456 + \\
&= 2100.90 \times .3456 + 1350.90 \times .1792 && = 2341.95 \\
E_4[v_2(x_1)] &= 4600.90 \times .1296 + 3664.49 \times .3456 + \\
&\quad 2600.90 \times .3456 + 2100.90 \times .1536 + \\
&\quad 1350.90 \times .0256 && = 3118.88
\end{aligned}$$

Values for $E[P_{x,y}] + E_y[v_2(x_1)]$. Production up to quantity for week 1

x	y=0	y=1	y=2	y=3	y=4
1		585.58 + 1448.10	353.49 + 1772.10	524.54 + 2341.95	184.00 + 3118.88

The optimum up to production for week 1: $y = 4$ and $E[P_{x,y}] = 3302.88$

(b) Allowing a maximum inventory of 5, the optimal policy is determined as follows: Compute $L(5) = 75 \times f_0 + 60 \times f_1 + 45 \times f_2 + 30 \times f_3 + 15 \times f_4 = 51.0$ Then add the following column to the table for the production up to quantity for week 3. (Note that a new row is added, but this row affects the new column only.)

x	y=5
0	$-\infty$
1	2720.00 - 3250 - 51
2	2720.00 - 2500 - 51
3	2720.00 - 2000 - 51
4	2720.00 - 1400 - 51
5	2720.00 - 0 - 51

The optimal values for the end of the second week are

x	0	1	2	3	4	5
v_3	24.54	585.58	1753.49	2524.54	2684.00	2669.00
y_3	3	1	2	3	4	5

The value for $E_5[v_3(x_2)]$ is given by:

$$E_5[v_3(x_2)] = 2669.00 \times .1296 + 2884.00 \times .3456 + 2524.54 \times .3456 + 1753.49 \times .1536 + 585.58 \times .0256 = 2499.42$$

Then add the following column to the table for the production up to quantity for week 2.

x	y=5
0	$-\infty$
1	-581.00 + 2499.42
2	169.00 + 2499.42
3	669.00 + 2499.42
4	1269.00 + 2499.42
5	26669.00 + 2499.42

Optimal values for the end of the first week:

x	0	1	2	3	4	5
v_2	1350.90	2100.90	2600.90	3664.49	4600.90	5168.42
y_2	4	4	5	3	4	5

The value for $E_5[v_2(x_1)]$ is given by:

$$E_5[v_2(x_1)] = 5168.42 \times .1296 + 4600.90 \times .3456 + 3664.49 \times .3456 + 2600.90 \times .1536 + 2100.90 \times .0256 = 3979.46$$

Then add the following column to the table for the production up to quantity for week 1.

x	y=5
5	-581.00 + 3979.63

The optimum up to production for week 1: $y = 5$ and $E[P_{x,y}] = 3398.63$.

(c) The simulation of this dynamic programming problem is somewhat difficult. What the student has to realize is the fact that the order up to quantity for a week depends on the ending inventory from the previous week (which is a random variable because the demand is a random variable). Thus, an order up to quantity cannot be specified a priori for every week before the simulation starts. The policy specified in the simulation will have the following characteristics: For week 1, specify the order up to quantity; for week 2, specify the order up to quantity given that the ending inventory from week 1 is 0, 1, 2, 3, or 4; for week 3, specify the order up to quantity given that the ending inventory from week 2 is 0, 1, 2, 3, or 4. The number of combinations is very large, and simulation is impractical in solving this problem.

10.15. Markov chain formulation for the multi period inventory problem with set-up costs.

(a) Let X_n be the inventory on hand at the end of a period. Then, the one-step transition probabilities for the (s,S) policy are:

$$P_{i,j} = \begin{cases} f(i-j) & \text{for } i > s, j > 0 \\ \sum_{k=i}^{\infty} f(k) & \text{for } i > s, j = 0 \\ f(S-j) & \text{for } i \le s, j > 0 \\ \sum_{k=S}^{\infty} f(k) & \text{for } i \le s, j = 0 \end{cases}$$

where $f(.)$ is the probability mass function for the demand.

(b) Markov matrix using a $(1,5)$ policy and a discrete uniform distribution between 0 and 3 units.

$$P = \begin{bmatrix} 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.50 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

(c) Ordering cost = 44.57
Storage cost = 18.375
Shortage cost = 5.75
The resulting long-run cost is \$68.695 per day.

(d) Keeping $S = 5$ fixed and evaluating for different values of s, the optimal policy is for $s = 1$ with a long-run cost of \$68.695. If the value of S can increase or decrease, the optimal (s,S) policy is still $s = 1$ and $S = 5$.

10.17. (a) $r = \lambda/\mu = 12/13$.

(b) $L = r = 12/13$.

(c)

$$P(Backorder) = 1 - 0.3973 - 0.3667 - 0.1693 = 0.0667$$

(d) The expected weekly cost is \$5714.03.

(e) The determine whether three items kept in the inventory if there are no orders being processed is optimal, evaluate the weekly cost for two items and four items. The optimal number turns out to be two items, as indicated in the following table.

Number of Items	0	1	2	3	4	5
Weekly Cost	5815	5735	5691	5714	5774	5845