Chapter 11
Replacement Theory

11.1. This problem is a repeat of the last problem of Chapter 1. It illustrates an important concept so that if it was not assigned as homework for Chapter 1, it may be worthwhile to assign as homework for this chapter.

$T \sim U(0,2)$, the cumulative distribution function for T is :

$$F(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{t}{2} & \text{for } 0 \leq t < 2 \\ 1 & \text{for } t \geq 2 \end{cases}$$

(a) $\Pr\{T \geq \frac{3}{12}\} = 1 - F(0.25) = 0.875$

(b) $\Pr\{T \geq 1.25 \mid T \geq 1\} = \frac{1-F(1.25)}{1-F(1)} = 0.75$

(c) If $T \sim Exp(1)$, the cumulative distribution fuction for T is :

$$G(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } t \geq 0 \end{cases}$$

$\Pr\{T \geq \frac{3}{12}\} = 1 - G(0.25) = 0.7788$

$\Pr\{T \geq 1.25 \mid T \geq 1\} = \frac{1-G(1.25)}{1-G(1)} = 0.7788$

(d) For the exponential distribution, it is not important to know how old of the machine is because of its memoryless property; however, for any other continuous distribution, it is important to know the machine's age, as is illustrated by the fact that the answers to part (a) and (b) differ.

11.3. (a) The optimal replacement time according to the following table is 14 months.

k	f_k	F_k	$c_r + c_f F_k$	$\overline{F_k}$	$\sum_{i=0}^{k-1} \overline{F_k}$	$z(k)$
1	0.05	0.05	4050	0.95	1.00	4050.00
2	0.03	0.08	4080	0.92	1.95	2092.31
3	0.02	0.10	4100	0.90	2.87	1428.57
4	0.00	0.10	4100	0.90	3.77	1087.53
5	0.00	0.10	4100	0.90	4.67	877.94
6	0.00	0.10	4100	0.90	5.57	736.09
7	0.00	0.10	4100	0.90	6.47	633.69
8	0.00	0.10	4100	0.90	7.37	556.31
9	0.00	0.10	4100	0.90	8.27	495.77
10	0.02	0.12	4120	0.88	9.17	449.29
11	0.03	0.15	4150	0.85	10.05	412.94
12	0.05	0.20	4200	0.80	10.90	385.32
13	0.10	0.30	4300	0.70	11.70	367.52
14	0.15	0.45	4450	0.55	12.40	358.87
15	0.25	0.70	4700	0.30	12.95	362.93

(b) Let u be the mean life of the battery, then the average cost z'_k for the policy that simply replaces at failure is :

$$z'_k = \frac{c_r + c_f}{u}$$

$\Rightarrow z'_k$ = 370.233

The average cost for the policy obtained from (a) is $z_k + 10$ = 368.87 ($<$ 370.233).

Therefore, the optimal replacement policy is the same as (a).

11.5. The simulation of Part (a) of Problem 11.4 is given in the Excel file associated with this chapter. To obtain the random life length of the battery, the CDF of the battery life is determined and then a `MATCH( )` function is used to give the mapping from the random number to the battery life. To keep the file size small, a full simulation is not given in the spreadsheet. You must extend the simulation down for several thousand rows. Hitting the F9 key causes the simulation to be re-run. (Note typo in the text. This problem refers to Problem 11.4, not Exercise 8.4.)

11.7. Show that

$$t_0 h(t_0) - \int_0^{t_0} h(s)ds = \frac{c_r}{c_m}$$

yields a unique solution to

$$\min_{t_0} z(t_0) = \frac{c_r + c_m \int_0^{t_0} h(s)ds}{t_0}$$

when the system has an IFR distribution.

Take the derivative of the left hand side of Eq. (8.6) and show that it is strictly increasing for all $t_0 > 0$.

$$\begin{aligned}\frac{d}{dt_0}\{t_0h(t_0) - \int_0^{t_0} h(s)ds\} &= t_0\frac{dh(t_0)}{dt_0} + h(t_0) - h(t_0)\\ &= \\ &= t_0\frac{dh(t_0)}{dt_0}\end{aligned}$$

For $t_0 > 0$, $h(.)$ is an increasing function (IFR), therefore, $dh(t_0)/dt_0 > 0$, and

$$t_0\frac{dh(t_0)}{dt_0} > 0$$

for all $t_0 > 0$. This implies that the solution for Eq. (8.5) given by Eq. (8.6) is unique.

11.9. Block replacement of 25 sensors, with setup cost $K = \$900$, replacement cost $c_r = 75$, and Poisson lifetime distribution with mean of 10 months. Optimal block replacement policy.

$$\min_i z(i) = \frac{1}{i}\{900 + 25 \times 75 + (900 + 75)\sum_{m=1}^{i} \bar{n}_m\}$$

or

$$\min_i z(i) = \frac{1}{i}\{2775 + 975\sum_{m=1}^{i} \bar{n}_m\}$$

The Poisson probabilities $p_1, p_2, \ldots, p_5$ are, respectively:

$$0.000454, 0.00227, 0.00757, 0.01892, 0.03783$$

For $i = 1$

$$\bar{n}_0 = 25, \ \bar{n}_1 = p_1 25 = 0.01135$$
$$z(1) = 2786.07$$

For $i = 2$

$$\bar{n}_2 = p_2 25 + p_1 0.01135 = 0.05675$$
$$z(2) = 1420.70$$

For $i = 3$

$$\bar{n}_3 = p_3 25 + p_2 0.01135 + p_1 0.05675 = 0.1893$$
$$z(3) = 1008.66$$

For $i = 4$

$$\bar{n}_4 = p_4 25 + p_3 0.01135 + p_2 0.05675 + p_1 0.1893 = 0.4733$$

$$z(4) = 871.86$$

For $i = 5$

$$\bar{n}_5 = p_5 25 + p_4 0.01135 + p_3 0.05675 + p_2 0.1893 + p_1 0.4733 = 0.9470$$

$$z(5) = 882.16$$

Because $z(5) > z(4)$, the optimal block replacement time is at the end of the fourth month, with an average replacement cost of \$871.86 per month.

Use Eq. (8.11) to check if it is advantageous to replace only failed items and never do block replacement.

$$z(\infty) = \frac{25 \times 975}{10} = \$2437.50$$

The average cost of never using preventive block replacement is larger than the average block replacement cost. Thus a block replacement policy is preferable.

11.11. From Eq. (8.10) we can see:

$$z(1) = K + \overline{n}_0 c_r + (K + c_r)\overline{n}_1,$$

or equivalently,

$$z(1) = K(1 + \overline{n}_1) + c_r(\overline{n}_0 + \overline{n}_1).$$

Also

$$\begin{aligned} z(2) &= \tfrac{1}{2}\{K + \overline{n}_0 c_r + (K + c_r)(\overline{n}_1 + \overline{n}_2)\} \\ &= \tfrac{1}{2}\{K + \overline{n}_0 c_r + (K + c_r)\overline{n}_1 + (K + c_r)\overline{n}_2\} \\ &= \tfrac{1}{2}\{z(1) + (K + c_r)\overline{n}_2\} \end{aligned}$$

Similarly, for $i = 3$,

$$\begin{aligned} z(3) &= \tfrac{1}{3}\{K + \overline{n}_0 c_r + (K + c_r)(\overline{n}_1 + \overline{n}_2 + \overline{n}_3)\} \\ &= \tfrac{1}{3}\{K + \overline{n}_0 c_r + (K + c_r)(\overline{n}_1 + \overline{n}_2) + (K + c_r)\overline{n}_3)\} \\ &= \tfrac{1}{3}\{2z(2) + (K + c_r)\overline{n}_3)\} \end{aligned}$$

In general,

$$\begin{aligned} z(i) &= \tfrac{1}{i}\{K + \overline{n}_0 c_r + (K + c_r)\textstyle\sum_{m=1}^{i}\overline{n}_m\} \\ &= \tfrac{1}{i}\{K + \overline{n}_0 c_r + (K + c_r)\textstyle\sum_{m=1}^{i-1}\overline{n}_m + (K + c_r)\overline{n}_i\} \\ &= \tfrac{1}{i}\{(i-1)z(i-1) + (K + c_r)\overline{n}_i\} \end{aligned}$$