Chapter 13
Advanced Queues

13.1. (a) $x_6 = 2 \times 29 + 12 = 70$
(b) $x_{10} = \frac{\sqrt{2}}{4}\left(1+\sqrt{2}\right)^{10} - \frac{\sqrt{2}}{4}\left(1-\sqrt{2}\right)^{10} = 2378$

13.3. The parameters are $\lambda = 1/\text{hr}$ and $\mu = 0.8284/\text{hr}$. (There will be some round-off error for this problem. The mean rate should actually be $2(\sqrt{2}-1)$.

(a) The generator is

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & 0 & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & 0 & -(\lambda+\mu) & \lambda & \cdots \\ 0 & 0 & \mu & 0 & -(\lambda+\mu) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

The characteristic equation is

$$\mu z^3 - (\lambda+\mu)z + \lambda = 0 .$$

(b) The equation factors into $(z-1)(\mu z^2 + \mu z - \lambda)$, and the quadratic part has roots given by

$$\begin{aligned} z &= \frac{-\mu \pm \sqrt{\mu^2 + 4\mu\lambda}}{2\mu} \\ &= \frac{-0.8284 \pm \sqrt{0.8284^2 + 4(0.8284)}}{2(0.8284)} \\ &= \frac{-0.8284 \pm 2}{1.6568} \end{aligned}$$

Thus, $z_1 = 0.707$ and $z_2 = -1.707$. Since the absolute value of z_2 is greater than 1, the value of c_2 must be zero or the norming equation would not converge.

Also note that the limits on the general equation is for $n \geq 2$, so the lowest indexed p included in the general equation is p_1; therefore, $p_n = c_1 z_1^n$ for $n \geq 1$. Redefining the constant yields $p_n = p_1(0.707)^{n-1}$.

(c) To find the values for p_0 and p_1, we first use the first equation from part (a):

$$\begin{aligned} -\lambda p_0 + \mu p_2 &= 0 \\ -p_0 + (0.8284)\cdot(0.707)p_1 &= 0 \\ p_0 &= 0.5857 p_1 \end{aligned}$$

Now using the norming equation:

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} p_n = p_0 + \sum_{n=1}^{\infty} p_n \\ &= 0.5857 p_1 + p_1 \sum_{n=1}^{\infty} 0.707^{n-1} \\ &= p_1 (0.5857 + \frac{1}{1-0.707}) = 4.0 \end{aligned}$$

Thus, $p_0 = 0.146$ and

$$p_n = \frac{0.707^{n-1}}{4} \text{ for } n = 1, 2, \cdots .$$

Finally,

$$\begin{aligned} P\{N > 3\} &= 1 - P\{N \leq 3\} \\ &= 1 - p_0 - p_1 - p_2 - p_3 \\ &= 1 - 0.146 - 0.25 - 0.178 - 0.125 = 0.301. \end{aligned}$$

(d) $E[N] = 2.91$

13.5. The generator matrix is

$$\mathbf{Q} = \left[\begin{array}{c|cc|cc|cc|c} -\lambda & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots \\ \hline \mu & -(\lambda+\mu) & \lambda_1 & \lambda_2 & 0 & 0 & 0 & \cdots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda_1 & \lambda_2 & 0 & 0 & \cdots \\ \hline 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda_1 & \lambda_2 & 0 & \cdots \\ 0 & 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda_1 & \lambda_2 & \cdots \\ \hline 0 & 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda_1 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 2\mu & -(\lambda+2\mu) & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] .$$

The system of equations that must be solved to obtain the steady-state probabilities is

$$
\begin{aligned}
p_0(\lambda_1,\lambda_2)+\mathbf{p}_1\begin{bmatrix}-(\lambda+\mu) & \lambda_1\\ 2\mu & -(\lambda+2\mu)\end{bmatrix}+\mathbf{p}_2\mathbf{M}&=\mathbf{0}\\
\mathbf{p}_{n-1}\boldsymbol{\Lambda}+\mathbf{p}_n\mathbf{A}+\mathbf{p}_{n+1}\mathbf{M}&=\mathbf{0}\ \text{ for } n=2,3,\cdots\\
p_0+\sum_{n=1}^{\infty}\mathbf{p}_n&=0
\end{aligned}
$$

where

$$
\boldsymbol{\Lambda}=\begin{bmatrix}\lambda_2 & 0\\ \lambda_1 & \lambda_2\end{bmatrix}\ \ \mathbf{A}=\begin{bmatrix}-(\lambda+2\mu) & \lambda_1\\ 2\mu & -(\lambda+2\mu)\end{bmatrix}\ \ \mathbf{M}=\begin{bmatrix}0 & 2\mu\\ 0 & 0\end{bmatrix}.
$$

Thus the matrix characteristic equation is

$$
\mathbf{R}^2\mathbf{M}+\mathbf{R}\mathbf{A}+\boldsymbol{\Lambda}=\mathbf{0}
$$

with the boundary conditions given by

$$
\begin{aligned}
p_0(\lambda_1,\lambda_2)+\mathbf{p}_1\begin{bmatrix}-(\lambda+\mu) & \lambda_1\\ 2\mu & -(\lambda+2\mu)\end{bmatrix}+\mathbf{p}_1\mathbf{R}\mathbf{M}&=\mathbf{0}\\
p_0+p_1(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1}&=0
\end{aligned}
$$

13.7. A state of the system will be defined by (n,v), where n is the total number of calls in the system and v is the number of voice calls is service. Note that since voice calls cannot wait, all jobs waiting are data calls.

(a) The matrix $\mathbf{Q}$ is defined (with $\lambda=\lambda_d+\lambda_v$ and $\times$ representing the negative of the row sum) by

$$
\begin{array}{c}
(0,0)\\(1,0)\\(1,1)\\ \hline (2,0)\\(2,1)\\(2,2)\\ \hline (3,0)\\(3,1)\\(3,2)\\ \hline \vdots
\end{array}
\left[\begin{array}{ccc|ccc|ccc|c}
-\lambda & \lambda_d & \lambda_v & & & & & & & \cdots\\
\mu_d & \times & & \lambda_d & \lambda_v & & & & & \cdots\\
\mu_v & & \times & & \lambda_d & \lambda_v & & & & \cdots\\ \hline
 & 2\mu_d & & \times & & & \lambda_d & & & \cdots\\
 & \mu_v & \mu_d & & \times & & & \lambda_d & & \cdots\\
 & & 2\mu_v & & & \times & & & \lambda_d & \cdots\\ \hline
 & & & 2\mu_d & & & \times & & & \cdots\\
 & & & \mu_v & \mu_d & & & \times & & \cdots\\
 & & & & 2\mu_v & & & & \times & \cdots\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

Thus the matrix characteristic equation is

$$
\mathbf{R}^2\mathbf{M}+\mathbf{R}\mathbf{A}+\boldsymbol{\Lambda}=\mathbf{0}
$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_d & 0 \\ 0 & 0 & \lambda_d \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 2\mu_d & 0 & 0 \\ \mu_v & \mu_d & 0 \\ 0 & 2\mu_v & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -(\lambda_d + 2\mu_d) & 0 & 0 \\ 0 & -(\lambda_d + \mu_d + \mu_v) & 0 \\ 0 & 0 & -(\lambda_d + 2\mu_v) \end{bmatrix} .$$

For the boundary conditions, partition the steady-state probability vector such that $\mathbf{p}_1 = (p_{00}, p_{10}, p_{11})$ and $\mathbf{p}_n = (p_{n0}, p_{n1}, p_{n2})$ for $n \geq 2$. Then the boundary conditions are

$$\mathbf{p}_1 \begin{bmatrix} -\lambda & \lambda_d & \lambda_v \\ \mu_d & -(\lambda + \mu_d) & 0 \\ \mu_v & 0 & -(\lambda + \mu_v) \end{bmatrix} + \mathbf{p}_2 \begin{bmatrix} 0 & 2\mu_d & 0 \\ 0 & \mu_v & \mu_d \\ 0 & 0 & 2\mu_v \end{bmatrix} = \mathbf{0}$$

$$\mathbf{p}_1 \begin{bmatrix} 0 & 0 & 0 \\ \lambda_d & \lambda_v & 0 \\ 0 & \lambda_d & \lambda_v \end{bmatrix} + \mathbf{p}_2\mathbf{A} + \mathbf{p}_2\mathbf{R}\mathbf{M} = \mathbf{0}$$

$$\mathbf{p}_1\mathbf{1} + \mathbf{p}_2 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{1} = 1$$

(b) The matrix equation to be solved is

$$\mathbf{R}^2 \begin{bmatrix} 120 & 0 & 0 \\ 5 & 60 & 0 \\ 0 & 10 & 0 \end{bmatrix} + \mathbf{R} \begin{bmatrix} -180 & 0 & 0 \\ 0 & -125 & 0 \\ 0 & 0 & -70 \end{bmatrix} + \begin{bmatrix} 60 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 60 \end{bmatrix} = \mathbf{0}$$

where

$$\mathbf{R} = \begin{bmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \text{ and } \mathbf{R}^2 = \begin{bmatrix} r_{11}^2 & 0 & 0 \\ r_{21}^{(2)} & r_{22}^2 & 0 \\ r_{31}^{(2)} & r_{32}^{(2)} & r_{33}^2 \end{bmatrix} .$$

We begin by looking at the three diagonal elements which yield the following system of equations

$$\begin{aligned} 120r_{11}^2 - 180r_{11} + 60 &= 0 \\ 60r_{22}^2 - 125r_{22} + 60 &= 0 \\ -70r_{33} + 60 &= 0 \end{aligned}$$

which implies $r_{11} = 0.5$, $r_{22} = 0.75$, and $r_{33} = 6/7$. (Notice that for the quadratic equaitons, we always took the root less than one.) The two equations that result from looking at the first subdiagonal yield

$$\begin{aligned} 120r_{21}^{(2)} + 5r_{22}^2 - 180r_{21} &= 0 \\ 60r_{32}^{(2)} + 10r_{33}^2 - 125r_{32} &= 0 . \end{aligned}$$

Because $r_{21}^{(2)} = r_{21}r_{11} + r_{22}r_{21}$ and $r_{32}^{(2)} = r_{32}r_{22} + r_{33}r_{32}$, it follows that the above is equivalent to

$$\begin{aligned} 120(r_{21}r_{11} + r_{22}r_{21}) + 5r_{22}^2 - 180r_{21} &= 0 \\ 60(r_{32}r_{22} + r_{33}r_{32}) + 10r_{33}^2 - 125r_{32} &= 0 \,. \end{aligned}$$

Thus, $r_{21} = 0.0938$ and $r_{32} = 0.2571$. The final equation is obtained by looking at the bottom left element in the above matrix equation; namely,

$$120r_{31}^{(2)} + 5r_{32}^{(2)} - 180r_{31} = 0$$

which is rewritten as

$$120(r_{31}r_{11} + r_{32}r_{21} + r_{33}r_{31}) + 5(r_{32}r_{22} + r_{33}r_{32}) - 180r_{31} = 0$$

Thus, $r_{31} = 0.2251$. In other words,

$$\mathbf{R} = \begin{bmatrix} 0.5000 & 0 & 0 \\ 0.0938 & 0.7500 & 0 \\ 0.2892 & 0.2571 & 0.8571 \end{bmatrix}$$

(c) First

$$\begin{aligned} \mathbf{p}_1 &= (\,0.191, 0.198, 0.061\,) \\ \mathbf{p}_2 &= (\,0.107, 0.056, 0.003\,)\,. \end{aligned}$$

$L_{data} = 2.7$.

(d) The probability that two voice calls are in the system is given by

$$\sum_{n=2}^{\infty} \mathbf{p}_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.024\,.$$

(e) The conditional expected number of data calls in the system given that voice calls occupy both channels is

$$L_{d|V=2} = 6.06$$

13.9. Current System: The M/M/1 system.
Since $\rho = 5/6$, we have $p_0 = 1/6$ and $L = 5$; thus

$$E[C] = 100(1 - p_0) + 50L = 333.33\,.$$

Alternative System: The M/E_4/1 system.

$$E[C] = 100(1 - p_0) + 50L = 289.33$$

thus, there is the potential to save $44 per hour.

13.11. A state of the system will be defined by (k,n), where k indicates the phase that the arriving customer is in and n is the total number of customers in the system. (The "arriving customer" is not yet in the system.)

(a) To find $\mathbf{R}$ analytically, first show that the matrix quadratic equation defining $\mathbf{R}$ can be written as

$$\begin{bmatrix} 0 & 0 \\ r_{21}r_{22} & r_{22}^2 \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} -(2\lambda+\mu) & 0 \\ 0 & -(2\lambda+\mu) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2\lambda & 0 \end{bmatrix} = \mathbf{0}\,.$$

which leads to the following system (after dividing both equations by μ and letting $\rho = \lambda/\mu$).

$$\begin{aligned} r_{21}r_{22} - (2\rho+1)r_{21} + 2\rho &= 0 \\ r_{22}^2 + 2\rho r_{21} - (2\rho+1)r_{22} &= 0\,. \end{aligned}$$

After a substitution, a cubic equation in r_{22} is obtained. Noting that $r_{22} = 1$ is a solution to the cubic equation, we can factor the cubic equation and obtain the following quadratic equation:

$$r_{22}^2 - (4\rho+1)r_{22} + 4\rho^2 = 0$$

which yields

$$\begin{aligned} r_{22} &= 2\rho + 0.5 - \sqrt{2\rho + 0.25} \\ r_{21} &= \left((2\rho+1)r_{22} - r_{22}^2\right)/(2\rho)\,. \end{aligned}$$

Using the fact that for this problem, $\rho = 10/12$, the $\mathbf{R}$ matrix is

$$\mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0.8844 & 0.7822 \end{bmatrix}.$$

(b) The probability that a queue is present

$$1 - 0.058 - 0.109 - 0.8844 \times 0.109 - 0.7822 \times 0.109 = 0.651\,.$$

(c) For the expected number in the system, we have

$$\begin{aligned} L &= \mathbf{p}_0\mathbf{R}(\mathbf{I}-\mathbf{R})^{-2}\mathbf{1} \\ &= (0.8844p_{20}, 0.7822p_{20})\,(1, 43.795)^T = 3.83\,. \end{aligned}$$

13.13. Let the arrival process have parameters $\mathbf{G}_*$, $\mathbf{G}_\Delta$, and α_*. The mean service rate is given by μ. The generator is thus given by

$$\mathbf{Q}=\begin{bmatrix} \mathbf{G}_* & \mathbf{G}_\Delta\alpha & & & & \cdots \\ \mu\mathbf{I} & \mathbf{G}_*-\mu\mathbf{I} & \mathbf{G}_\Delta\alpha & & & \cdots \\ & 2\mu\mathbf{I} & \mathbf{G}_*-2\mu\mathbf{I} & \mathbf{G}_\Delta\alpha & & \cdots \\ & & 2\mu\mathbf{I} & \mathbf{G}_*-2\mu\mathbf{I} & \mathbf{G}_\Delta\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix characteristic equation is

$$2\mu\mathbf{R}^2+\mathbf{R}(\mathbf{G}_*-2\mu\mathbf{I})+\mathbf{G}_\Delta\alpha=\mathbf{0}$$

which implies that $\mathbf{p}_n = p_1\mathbf{R}^{n-1}$ for $n=1,2,\cdots$. The boundary equations are

$$\begin{aligned} \mathbf{p}_0\mathbf{G}_*+\mu\mathbf{p}_1 &= \mathbf{0} \\ \mathbf{p}_0\mathbf{G}_\Delta\alpha+\mathbf{p}_1(\mathbf{G}_*-\mu\mathbf{I}+2\mu\mathbf{R}) &= \mathbf{0} \\ \mathbf{p}_0\mathbf{1}+\mathbf{p}_1(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1} &= 1\,. \end{aligned}$$

It should also be noted that the vector multiplication $\mathbf{G}_\Delta\alpha$ results in a matrix, since the first vector is a column vector and the second one is a row vector.