Chapter 13
Advanced Queues

13.1. (a) $x_6 = 2 \times 29 + 12 = 70$
(b) $x_{10} = \sqrt{2} \left(1 + \sqrt{2}\right)^{10} - \sqrt{2} \left(1 - \sqrt{2}\right)^{10} = 2387$

13.3. The parameters are $\lambda = 1/hr$ and $\mu = 0.8284/hr$. (There will be some round-off error for this problem. The mean rate should actually be $2(\sqrt{2} - 1)$.)

(a) The generator is

$$Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & 0 & \cdots \\
0 & -\lambda & \lambda & 0 & 0 & \cdots \\
\mu & 0 & -\left(\lambda + \mu\right) & \lambda & 0 & \cdots \\
0 & \mu & 0 & -\left(\lambda + \mu\right) & \lambda & \cdots \\
0 & 0 & \mu & 0 & -\left(\lambda + \mu\right) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}$$

The characteristic equation is

$$\mu z^3 - (\lambda + \mu)z + \lambda = 0.$$ 

(b) The equation factors into $(z - 1)(\mu z^2 + \mu z - \lambda)$, and the quadratic part has roots given by

$$z = \frac{-\mu \pm \sqrt{\mu^2 + 4\mu \lambda}}{2\mu}$$

$$= \frac{-0.8284 \pm \sqrt{0.8284^2 + 4(0.8284)}}{2(0.8284)}$$

$$= \frac{-0.8284 \pm 2}{1.6568}$$

Thus, $z_1 = 0.707$ and $z_2 = -1.707$. Since the absolute value of $z_2$ is greater than 1, the value of $c_2$ must be zero or the norming equation would not converge.
Also note that the limits on the general equation is for \( n \geq 2 \), so the lowest indexed \( p \) included in the general equation is \( p_1 \); therefore, \( p_n = c_1 z_1^n \) for \( n \geq 1 \). Redefining the constant yields \( p_n = p_1 (0.707)^{n-1} \).

(c) To find the values for \( p_0 \) and \( p_1 \), we first use the first equation from part (a):

\[
-\lambda p_0 + \mu p_2 = 0 \\
-p_0 + (0.8284) \cdot (0.707)p_1 = 0 \\
p_0 = 0.5857 p_1
\]

Now using the norming equation:

\[
1 = \sum_{n=0}^{\infty} p_n = p_0 + \sum_{n=1}^{\infty} p_n \\
= 0.5857 p_1 + p_1 \sum_{n=1}^{\infty} 0.707^{n-1} \\
= p_1 (0.5857 + \frac{1}{1-0.707}) = 4.0
\]

Thus, \( p_0 = 0.146 \) and

\[
p_n = \frac{0.707^{n-1}}{4} \quad \text{for } n = 1, 2, \ldots
\]

Finally,

\[
P\{N > 3\} = 1 - P\{N \leq 3\} \\
= 1 - p_0 - p_1 - p_2 - p_3 \\
= 1 - 0.146 - 0.25 - 0.178 - 0.125 = 0.301.
\]

(d) \( E[N] = 2.91 \)

13.5. The generator matrix is

\[
Q = \begin{bmatrix}
-\lambda & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots \\
\mu & -(\lambda + \mu) & \lambda_1 & \lambda_2 & 0 & 0 & 0 & \cdots \\
0 & 2\mu & -(\lambda + 2\mu) & \lambda_1 & \lambda_2 & 0 & 0 & \cdots \\
0 & 0 & 2\mu & -(\lambda + 2\mu) & \lambda_1 & \lambda_2 & \cdots & \cdots \\
0 & 0 & 0 & 2\mu & -(\lambda + 2\mu) & \lambda_1 & \cdots & \cdots \\
0 & 0 & 0 & 0 & 2\mu & -(\lambda + 2\mu) & \lambda_1 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots
\end{bmatrix}
\]

The system of equations that must be solved to obtain the steady-state probabilities is
Advanced Queues

\[ p_0(\lambda_1, \lambda_2) + p_1 \left[ \begin{array}{cc} -(\lambda + \mu) & \lambda_1 \\ 2\mu & -(\lambda + 2\mu) \end{array} \right] + p_2 M = 0 \]

\[ p_{n-1} A + p_n A + p_{n+1} M = 0 \text{ for } n = 2, 3, \ldots \]

\[ p_0 + \sum_{n=1}^{\infty} p_n = 0 \]

where

\[ A = \begin{bmatrix} \lambda_2 & 0 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad A = \begin{bmatrix} -(\lambda + 2\mu) & \lambda_1 \\ 2\mu & -(\lambda + 2\mu) \end{bmatrix} \quad M = \begin{bmatrix} 0 & 2\mu \\ 0 & 0 \end{bmatrix} . \]

Thus the matrix characteristic equation is

\[ R^2 M + RA + A = 0 \]

with the boundary conditions given by

\[ p_0(\lambda_1, \lambda_2) + p_1 \left[ \begin{array}{cc} -(\lambda + \mu) & \lambda_1 \\ 2\mu & -(\lambda + 2\mu) \end{array} \right] + p_1 R M = 0 \]

\[ p_0 + p_1 (I - R)^{-1} 1 = 0 \]

13.7. A state of the system will be defined by \((n, v)\), where \(n\) is the total number of calls in the system and \(v\) is the number of voice calls in service. Note that since voice calls cannot wait, all jobs waiting are data calls.

(a) The matrix \(Q\) is defined (with \(\lambda = \lambda_d + \lambda_v\) and \(\times\) representing the negative of the row sum) by

\[
\begin{array}{cccccc}
(0, 0) & [-(\lambda + \mu) & \lambda_d & \lambda_v] & \cdots \\
(1, 0) & \mu_d \times & \lambda_d & \lambda_v & \cdots \\
(1, 1) & \mu_v \times & \lambda_d & \lambda_v & \cdots \\
(2, 0) & 2\mu_d \times & \lambda_d & \cdots \\
(2, 1) & \mu_v & \mu_d \times & \lambda_d & \cdots \\
(2, 2) & 2\mu_v \times & \lambda_d & \cdots \\
(3, 0) & 2\mu_d \times & \cdots \\
(3, 1) & \mu_v & \mu_d \times & \cdots \\
(3, 2) & 2\mu_v \times & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

Thus the matrix characteristic equation is

\[ R^2 M + RA + A = 0 \]

where
\[
A = \begin{bmatrix}
\lambda_d & 0 & 0 \\
0 & \lambda_d & 0 \\
0 & 0 & \lambda_d
\end{bmatrix}, \\
M = \begin{bmatrix}
2\mu_d & 0 & 0 \\
\mu_v & \mu_d & 0 \\
0 & 2\mu_v & 0
\end{bmatrix}
\]
\[
\mathbf{M} = \begin{bmatrix}
2\mu_d \\
\mu_v \\
0
\end{bmatrix}
\]
\[
A = \begin{bmatrix}
-(\lambda_d + 2\mu_d) & 0 & 0 \\
0 & -(\lambda_d + \mu_d + \mu_v) & 0 \\
0 & 0 & -(\lambda_d + 2\mu_v)
\end{bmatrix}.
\]

For the boundary conditions, partition the steady-state probability vector such that
\[
p_1 = (p_{00}, p_{10}, p_{11}) \quad \text{and} \quad p_n = (p_{n0}, p_{n1}, p_{n2}) \quad \text{for} \quad n \geq 2.
\]

Then the boundary conditions are
\[
p_1 \begin{bmatrix}
-\lambda \\
\mu_d \\
\mu_v
\end{bmatrix} + \begin{bmatrix}
\lambda_d \\
0 \\
0
\end{bmatrix} + p_2 \begin{bmatrix}
0 & 0 & 0 \\
0 & \mu_v & \mu_d \\
0 & 0 & 2\mu_v
\end{bmatrix} = 0
\]
\[
p_1 \begin{bmatrix}
\lambda_d \\
\lambda_v \\
\lambda_d
\end{bmatrix} + p_2 A + p_2 \mathbf{M} = 0
\]
\[
p_1 \textbf{1} + p_2 (\textbf{I} - \textbf{R})^{-1} \textbf{1} = 1
\]

(b) The matrix equation to be solved is
\[
\mathbf{R}^2 \begin{bmatrix}
120 & 0 & 0 \\
5 & 60 & 0 \\
0 & 10 & 0
\end{bmatrix} + \mathbf{R} \begin{bmatrix}
-180 & 0 & 0 \\
0 & -125 & 0 \\
0 & 0 & -70
\end{bmatrix} + \begin{bmatrix}
60 & 0 & 0 \\
0 & 60 & 0 \\
0 & 0 & 60
\end{bmatrix} = \mathbf{0}
\]

where
\[
\mathbf{R} = \begin{bmatrix}
r_{11} & 0 & 0 \\
r_{21} & r_{22} & 0 \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} \quad \text{and} \quad \mathbf{R}^2 = \begin{bmatrix}
r_{11}^2 & 0 & 0 \\
r_{21}^2 & r_{22}^2 & 0 \\
r_{31}^2 & r_{32}^2 & r_{33}^2
\end{bmatrix}.
\]

We begin by looking at the three diagonal elements which yield the following system of equations
\[
120r_{11}^2 - 180r_{11} + 60 = 0
\]
\[
60r_{22}^2 - 125r_{22} + 60 = 0
\]
\[
-70r_{33} + 60 = 0
\]

which implies \(r_{11} = 0.5, r_{22} = 0.75, \) and \(r_{33} = 6/7\). (Notice that for the quadratic equations, we always took the root less than one.) The two equations that result from looking at the first subdiagonal yield
\[
120r_{21}^{(2)} + 5r_{22}^2 - 180r_{21} = 0
\]
\[
60r_{32}^{(2)} + 10r_{33}^2 - 125r_{32} = 0.
\]
Because \( r_{21}^{(2)} = r_{21}r_{11} + r_{22}r_{21} \) and \( r_{32}^{(2)} = r_{32}r_{22} + r_{33}r_{32} \), it follows that the above is equivalent to

\[
120(r_{21}r_{11} + r_{22}r_{21}) + 5r_{22} - 180r_{21} = 0 \\
60(r_{32}r_{22} + r_{33}r_{32}) + 10r_{33} - 125r_{32} = 0 .
\]

Thus, \( r_{21} = 0.0938 \) and \( r_{32} = 0.2571 \). The final equation is obtained by looking at the bottom left element in the above matrix equation; namely,

\[
120r_{31}^{(2)} + 5r_{32}^{(2)} - 180r_{31} = 0
\]

which is rewritten as

\[
120(r_{31}r_{11} + r_{32}r_{21} + r_{33}r_{31}) + 5(r_{32}r_{22} + r_{33}r_{32}) - 180r_{31} = 0
\]

Thus, \( r_{31} = 0.2251 \). In other words,

\[
R = \begin{bmatrix}
0.5000 & 0 & 0 \\
0.0938 & 0.7500 & 0 \\
0.2892 & 0.2571 & 0.8571
\end{bmatrix}
\]

(c) First

\[
p_1 = (0.191, 0.198, 0.061) \\
p_2 = (0.107, 0.056, 0.003).
\]

\( L_{\text{data}} = 2.7 \).

(d) The probability that two voice calls are in the system is given by

\[
\sum_{n=2}^{\infty} p_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.024.
\]

(e) The conditional expected number of data calls in the system given that voice calls occupy both channels is

\[
L_{d|V=2} = 6.06
\]

13.9. Current System: The M/M/1 system.

Since \( \rho = 5/6 \), we have \( p_0 = 1/6 \) and \( L = 5 \); thus

\[
E[C] = 100(1 - p_0) + 50L = 333.33.
\]

Alternative System: The M/E_4/1 system.

\[
E[C] = 100(1 - p_0) + 50L = 289.33
\]

thus, there is the potential to save $44 per hour.
13.11. A state of the system will be defined by \((k, n)\), where \(k\) indicates the phase that the arriving customer is in and \(n\) is the total number of customers in the system. (The “arriving customer” is not yet in the system.)

(a) To find \(R\) analytically, first show that the matrix quadratic equation defining \(R\) can be written as

\[
\begin{bmatrix}
  0 & 0 \\
  r_{21}r_{22} & r_{22}^2
\end{bmatrix}
\begin{bmatrix}
  \mu & 0 \\
  0 & \mu
\end{bmatrix}
+ \begin{bmatrix}
  0 & 0 \\
  r_{21} & r_{22}
\end{bmatrix}
\begin{bmatrix}
  -(2\lambda + \mu) & 0 \\
  0 & -(2\lambda + \mu)
\end{bmatrix}
+ \begin{bmatrix}
  0 & 0 \\
  2\lambda & 0
\end{bmatrix}
= 0 .
\]

which leads to the following system (after dividing both equations by \(\mu\) and letting \(\rho = \lambda/\mu\)).

\[
\begin{align*}
  r_{21}r_{22} - (2\rho + 1)r_{21} + 2\rho &= 0 \\
  r_{22}^2 + 2\rho r_{21} - (2\rho + 1)r_{22} &= 0 .
\end{align*}
\]

After a substitution, a cubic equation in \(r_{22}\) is obtained. Noting that \(r_{22} = 1\) is a solution to the cubic equation, we can factor the cubic equation and obtain the following quadratic equation:

\[
 r_{22}^2 - (4\rho + 1)r_{22} + 4\rho^2 = 0 .
\]

which yields

\[
\begin{align*}
  r_{22} &= 2\rho + 0.5 - \sqrt{2\rho + 0.25} \\
  r_{21} &= ((2\rho + 1)r_{22} - r_{22}^2) / (2\rho) .
\end{align*}
\]

Using the fact that for this problem, \(\rho = 10/12\), the \(R\) matrix is

\[
\begin{bmatrix}
  0 & 0 \\
  0.8844 & 0.7822
\end{bmatrix} .
\]

(b) The probability that a queue is present

\[
1 - 0.058 - 0.109 - 0.8844 \times 0.109 - 0.7822 \times 0.109 = 0.651 .
\]

(c) For the expected number in the system, we have

\[
L = p_0 R (I - R)^{-2} 1
= (0.8844 p_{20}, 0.7822 p_{20}) (1, 43.795)^T = 3.83 .
\]

13.13. Let the arrival process have parameters \(G_\ast, \ G_\Delta\), and \(\alpha_\ast\). The mean service rate is given by \(\mu\). The generator is thus given by
The matrix characteristic equation is

\[ 2\mu R^2 + R(G_s - 2\mu I) + G_\Delta \alpha = 0 \]

which implies that \( p_n = p_1 R^{n-1} \) for \( n = 1, 2, \cdots \). The boundary equations are

\[
\begin{align*}
p_0 G_s + \mu p_1 &= 0 \\
p_0 G_\Delta \alpha + p_1 (G_s - \mu I + 2\mu R) &= 0 \\
p_0 1 + p_1 (I - R)^{-1} 1 &= 1 .
\end{align*}
\]

It should also be noted that the vector multiplication \( G_\Delta \alpha \) results in a matrix, since the first vector is a column vector and the second one is a row vector.