Chapter 6
Markov Processes

6.1.

$$\mathbf{G} = \begin{bmatrix} -\mathbf{10} & 3 & 0 & 7 \\ 5 & -12 & 4 & \mathbf{3} \\ 1 & \mathbf{0} & -6 & 5 \\ 3 & 2 & 9 & -\mathbf{14} \end{bmatrix}$$

6.3. The events $\{U > u\}$ and $\{T > u\} \cap \{S > u\}$ are equal. To show this equality, observe that

$$\{U > u\} \Rightarrow \{T > u\} \cap \{S > u\} \text{ and}$$
$$\{T > u\} \cap \{S > u\} \Rightarrow \{U > u\} .$$

Since T and S are independent, the probabilities are multiplied.
Therefore, $P\{U > u\} = \mathrm{e}^{-au}\mathrm{e}^{-bu} = \mathrm{e}^{-(a+b)u}$.

6.5.

$$\mathbf{h} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5000 & 0 & 0 & 0 \end{bmatrix}$$

The cost rate vector $\mathbf{f} = (100, 300, 500, 1000)$ and $\mathbf{p} = (\frac{2.5}{10.75}, \frac{1}{10.75}, \frac{4}{10.75}, \frac{3.25}{10.75})$

$$\sum_{i \in E} p(i)\,[f(i) + \sum_{k \in E} G(i,k)\,h(i,k)] = 3563$$

After adding the maintenance, the new cost rate vector $\mathbf{f}' = (450, 550, 650, 900)$

$$\sum_{i \in E} p(i)\, f'(i) = 669.8\,(< 3563\,)$$

The maintenance cost is clearly worthwhile.

6.7.

$$\mathbf{G} = \begin{bmatrix} -10 & 1 & 7 & 2 & 0 & 0 \\ 6 & -16 & 1 & 7 & 2 & 0 \\ 0 & 12 & -22 & 1 & 7 & 2 \\ 0 & 0 & 18 & -26 & 1 & 7 \\ 0 & 0 & 0 & 24 & -25 & 1 \\ 0 & 0 & 0 & 0 & 30 & -30 \end{bmatrix}$$

6.9. (a) Students sometimes have difficulty with the state space; they often want to make "firing" a state.
$E = \{0, 1, 2, r\}$

$$\mathbf{G} = \begin{bmatrix} -90 & 90 & 0 & 0 \\ 0 & -90 & 90 & 0 \\ 0 & 0 & -90 & 90 \\ 60 & 0 & 0 & -60 \end{bmatrix}$$

(b) $p(r) = 0.333$
(c) To fire, the component needs to be in state 2 and then an impluse must arrive; therefore, $p(2) \times 90\,/hr = \frac{2}{9} \times 90\,/hr = 20\,/hr$

6.11. Let the generator be

$$\mathbf{G} = \begin{bmatrix} -\lambda_a & \lambda_a \\ \lambda_b & -\lambda_b \end{bmatrix}$$

The matrices for the diagonalization of the generator are

$$\mathbf{Q} = \begin{bmatrix} 1 & \lambda_a \\ 1 & -\lambda_b \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & -(\lambda_a + \lambda_b) \end{bmatrix}; \quad \mathbf{Q}^{-1} = \frac{1}{\lambda_a + \lambda_b} \begin{bmatrix} \lambda_b & \lambda_a \\ 1 & -1 \end{bmatrix}.$$

Using the equation $\mathrm{e}^{\mathbf{G}t} = \mathbf{Q}\,\mathrm{e}^{\mathbf{D}t}\,\mathbf{Q}^{-1}$ yields

$$\mathrm{e}^{\mathbf{G}t} = \frac{1}{\lambda_a + \lambda_b} \begin{bmatrix} \lambda_b + \lambda_a\,\mathrm{e}^{-(\lambda_a+\lambda_b)t} & \lambda_a - \lambda_a\,\mathrm{e}^{-(\lambda_a+\lambda_b)t} \\ \lambda_b - \lambda_b\,\mathrm{e}^{-(\lambda_a+\lambda_b)t} & \lambda_a + \lambda_b\,\mathrm{e}^{-(\lambda_a+\lambda_b)t} \end{bmatrix}.$$

Therefore

$$\Pr\{Y_t = a | Y_0 = a\} = \frac{\lambda_b}{\lambda_a + \lambda_b} + \frac{\lambda_a}{\lambda_a + \lambda_b}\,\mathrm{e}^{-(\lambda_a+\lambda_b)t}.$$

6.13.

$$\sum_{j \in E} f(j) \int_t^{t+s} \mathrm{e}^{-\beta u} \mathrm{e}^{u\mathbf{G}}(i,j)\,du$$